Homework Set #7

assigned: Friday, January 6th, 2017
due: Friday, January 20th, 2016, 3 pm
drop boxes outside 374 Firestone

Problem 1: averaging theorems (50 points).

We showed in class that volume averages can also be computed from information on the boundary via

$$\langle \boldsymbol{\varepsilon} \rangle = \frac{1}{V} \int_{\partial\Omega} \operatorname{sym}(\boldsymbol{u} \otimes \boldsymbol{n})\, \mathrm{d}S \approx \frac{1}{V} \sum_{a=1}^{n} \operatorname{sym}(\boldsymbol{u}^a \otimes \tilde{\boldsymbol{n}}^a) \qquad \text{with} \qquad \tilde{\boldsymbol{n}}^a = \frac{1}{2} \sum_e \boldsymbol{n}_e^a A_e$$

and

$$\langle \boldsymbol{\sigma} \rangle = \frac{1}{V} \int_{\partial\Omega} \boldsymbol{t} \otimes \boldsymbol{x}\, \mathrm{d}S \approx \frac{1}{V} \sum_a \boldsymbol{F}^a \otimes \boldsymbol{x}^a,$$

where summation over a indicates the sum over all boundary nodes (see *lecture notes*). $\boldsymbol{n}_e^a$ is the element outward normal vector at node a seen from element e (with surface area or length A_e). $\boldsymbol{F}^a$ is the total force at node a. We use linearized kinematics for convenience.

Let us verify those relations computationally, using our finite element code.

To this end, construct a square-shaped 2D RVE (use, e.g., CST elements) and a linear elastic material model in 2D. Create two different material models which differ in their moduli. Next, create a composite by randomly assigning one of two materials to 50% of the elements in your RVE. Apply *affine displacement BCs*

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{\varepsilon}_0 \boldsymbol{x} \qquad \text{for} \quad \boldsymbol{x} \in \partial\Omega$$

to every node on the outer boundary of the RVE for some deformation gradient $\boldsymbol{\varepsilon}_0 \neq \mathbf{0}$. Solve the boundary value problem (using, e.g., the Newton-Raphson solver).

Once you have the solution, let us verify the two above averaging theorems as follows:

1. Use methods `computeStressesAtGaussPoints` and `computeStrainsAtGaussPoints` (these exist inside the isoparametric element) to compute the averages $\langle \boldsymbol{\varepsilon} \rangle$ and $\langle \boldsymbol{\sigma} \rangle$ by summing over all elements in your FE mesh. Verify that indeed $\langle \boldsymbol{\varepsilon} \rangle = \boldsymbol{\varepsilon}_0$.
2. Compute $\langle \boldsymbol{\varepsilon} \rangle$ by using the above formula, summing over all boundary nodes.
3. Compute $\langle \boldsymbol{\sigma} \rangle$ by using the above formula, summing over all boundary nodes.

Hint: Instead of computing boundary normals in a general fashion, simply exploit the fact that your RVE is a square, so that you can sort boundary nodes by one of the four boundaries, on each of which the normal is known. Also, all element surface areas/lengths and all element volumes are known and the same in a regular mesh.

Problem 2: representative volume elements (50 points).

Let us use the same setup as above and try to derive the effective stiffness tensor of the composite.

The stiffness tensor $\widetilde{\mathbb{C}}^*$ in Voigt notation can be obtained from the linear system of equations

$$\langle\widetilde{\boldsymbol{\sigma}}\rangle = \widetilde{\mathbb{C}}^* \widetilde{\boldsymbol{\varepsilon}}_0.$$

Note that in 2D these are only 3 equations for the 9 unknown components of $\widetilde{\mathbb{C}}^*$ (one cannot a-priori assume symmetry of $\mathbb{C}^*$). Therefore, pick three different (linearly independent) vectors $\widetilde{\boldsymbol{\varepsilon}}_0 \neq \mathbf{0}$ (since the system is linear, the choice of $\boldsymbol{\varepsilon}_0$ is irrelevant) and for each choice compute $\langle\widetilde{\boldsymbol{\sigma}}\rangle$. Use Eigen or Matlab or Mathematica to compute the components of $\widetilde{\mathbb{C}}^*$ from the three computed $\langle\widetilde{\boldsymbol{\sigma}}\rangle$-vectors and the known corresponding three $\widetilde{\boldsymbol{\varepsilon}}_0$-vectors by solving the above linear system of equations.

Let us perform the following case study:

(i) Use the above setup to compute $\widetilde{\mathbb{C}}^*$ for a fixed number of elements n_e but by averaging over an increasing number N of random RVE realizations (i.e., simply re-run the analysis with a new random seed for the material model assignments). By plotting the mean and standard deviation of, e.g., $\widetilde{\mathbb{C}}_{11}$ vs. n_e show the convergence of $\widetilde{\mathbb{C}}^*$ with *ensemble enlargement*.

(ii) Repeat the above with finer meshes to show the effect to *sample enlargement*.

Hint: Note that you can automatize the re-running of the above analysis by having a `for`-loop around your entire `Main` routine and outputting the average stresses or moduli into files with reasonably-changing file names. No need to run each realization or simulation by hand.

total: 100 points