Homework Set #6

assigned: Monday, November 23rd, 2016
due: Tuesday, December 6th, 2016 (**no extensions**), 3 pm
drop boxes outside 374 Firestone

(Coding) Problem 1: inelastic material model (25 points).

Let us implement the viscoelastic generalized Maxwell model in linearized kinematics. As discussed in class, the (condensed) energy density is

$$W^*_{\{\boldsymbol{e}_p^{i,\alpha}\}}(\boldsymbol{\varepsilon}^{\alpha+1}) = \mathcal{F}_{\{\boldsymbol{e}_p^{i,\alpha}\}}(\boldsymbol{\varepsilon}^{\alpha+1}, \boldsymbol{e}_{p,*}^{1,\alpha+1}, \dots, \boldsymbol{e}_{p,*}^{n,\alpha+1})$$

with the incremental energy density

$$\mathcal{F}_{\{\boldsymbol{e}_p^{i,\alpha}\}}(\boldsymbol{\varepsilon}^{\alpha+1}, \boldsymbol{e}_p^{1,\alpha+1}, \dots, \boldsymbol{e}_p^{n,\alpha+1}) = \frac{\kappa_\infty}{2}(\operatorname{tr}\boldsymbol{\varepsilon}^{\alpha+1})^2 + \mu_\infty\, \boldsymbol{e}^{\alpha+1}\cdot\boldsymbol{e}^{\alpha+1} + \sum_{i=1}^n \mu_i \left\|\boldsymbol{e}^{\alpha+1} - \boldsymbol{e}_p^{i,\alpha+1}\right\|^2 + \sum_{i=1}^n \frac{\eta_i}{2\Delta t}\left\|\boldsymbol{e}_p^{i,\alpha+1} - \boldsymbol{e}_p^{i,\alpha}\right\|^2$$

whose minimization with respect to the new internal variables yields

$$\boldsymbol{e}_{p,*}^{i,\alpha+1} = \boldsymbol{e}_p^{i,\alpha} + \frac{2}{2+\tau_i/\Delta t}\left(\boldsymbol{e}^{\alpha+1} - \boldsymbol{e}_p^{i,\alpha}\right)$$

with relaxation time $\tau_i = \eta_i/\mu_i$. Insertion and differentiation leads to the stress tensor

$$\boldsymbol{\sigma}^{\alpha+1}(\boldsymbol{\varepsilon}^{\alpha+1}, \boldsymbol{e}_p^{1,\alpha}, \dots, \boldsymbol{e}_p^{n,\alpha}) = \kappa_\infty(\operatorname{tr}\boldsymbol{\varepsilon}^{\alpha+1})\boldsymbol{I} + 2\mu_\infty \boldsymbol{e}^{\alpha+1} + \sum_{i=1}^n 2\mu_i \frac{\tau_i/\Delta t}{2+\tau_i/\Delta t}\left(\boldsymbol{e}^{\alpha+1} - \boldsymbol{e}_p^{i,\alpha}\right).$$

and the incremental tangent matrix

$$\mathbb{C}_{ijkl}^{\alpha+1} = \left[\kappa_\infty - \frac{2}{3}\left(\mu_\infty + \sum_{i=1}^n \mu_i \frac{\tau_i/\Delta t}{2+\tau_i/\Delta t}\right)\right]\delta_{ij}\delta_{kl} + \left(\mu_\infty + \sum_{i=1}^n \mu_i \frac{\tau_i/\Delta t}{2+\tau_i/\Delta t}\right)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Notice that the ratio $\tau_i/\Delta t$ can be treated like one material/loading parameter (τ_i and Δt only appear in this combination).

Let us implement this material model in analogous fashion to the elastic material models (file `ViscoElastic.h`). The material model must define and its methods depend on `InternalVariables` $\boldsymbol{z} = \{\boldsymbol{e}_p^1, \dots, \boldsymbol{e}_p^n\}$. Use the derivative tester to verify the implementation of your material model.

(Coding) Problem 2: viscoelastic indentation BVP (15 points).

Let us apply the viscoelastic model to a quasistatic indentation test (using the simplicial element from set #5). We simulate a 3D cuboidal block (side length $a = 1$, e.g., 6 elements per side) made of a viscoelastic material ($\kappa_\infty = 10$, $\mu_\infty = 1$, $n = 3$ Maxwell elements with $\mu_i = \{0.1, 0.2, 0.3\}$ and $\tau_i = \{1.25, 2.0, 3.2\}$) whose bottom is fixed while a circular indenter (from problem set #5, force constant $C = 5\cdot 10^2$) is forced gradually into the top face at constant indentation rates. We can compute the total indenter force, e.g., by calculating the total force across all bottom nodes. Plot the *indenter force vs. indentation depth* for rates of $2.5\cdot 10^{-1}$, $2.5\cdot 10^{-2}$, $2.5\cdot 10^{-3}$ to demonstrate rate dependence (e.g., 200 load steps up to 0.5 indentation).

(Coding) Problem 3: dynamic solver (35 points).

In order to simulate the influence of inertia, we need a dynamic solver. Let us implement the *explicit dynamic solver* discussed in class, which computes the new displacements $\boldsymbol{U}^{\alpha+1}$ from the known previous values:

$$\left(\frac{\boldsymbol{M}}{\Delta t^2}+\frac{\boldsymbol{C}}{2\Delta t}\right)\boldsymbol{U}^{\alpha+1}=2\frac{\boldsymbol{M}}{\Delta t^2}\boldsymbol{U}^{\alpha}-\left(\frac{\boldsymbol{M}}{\Delta t^2}-\frac{\boldsymbol{C}}{2\Delta t}\right)\boldsymbol{U}^{\alpha-1}-\boldsymbol{F}_{\mathsf{int}}(\boldsymbol{U}^{\alpha},t^{\alpha}),$$

where we assumed that $\boldsymbol{F}_{\mathsf{int}}$ includes forces from internal and external elements. Thus, at each time step we compute $\boldsymbol{U}^{\alpha+1}$ based on the previously computed values of $\boldsymbol{U}^{\alpha}$ and $\boldsymbol{U}^{\alpha-1}$. We define $\boldsymbol{C}=\alpha\boldsymbol{M}+\beta\boldsymbol{K}$.

Hint: We have modified the element and assembler for you so they can provide a `LumpedMassMatrix` and `ConsistentMassMatrix` in complete analogy to the `StiffnessMatrix` (same functionality).

Note that *essential BCs* must be enforced, e.g., by replacing columns in the above system of equations like in the Newton-Raphson (NR) solver – feel free to reuse code from the NR solver. The above explicit dynamics method is implemented in the new function `computeUpdatedDisplacements` (see file `ExplicitDynamics.h`).

(Coding) Problem 4: wall potential (10 points).

To simplify contact problems, let us implement the potential of a stiff planar wall in close analogy to the indenter potential from before (see file `Wall.h`). Here, the potential energy for a point $\boldsymbol{x}_i$ and a wall going through a known point $\boldsymbol{x}_0$ and having unit normal $\boldsymbol{n}$ (into the direction of the impacting object) is

$$I_e=\frac{k}{2}\left\|(\boldsymbol{x}_0-\boldsymbol{x}_i)\cdot\boldsymbol{n}\right\|^3 H\big((\boldsymbol{x}_0-\boldsymbol{x}_i)\cdot\boldsymbol{n}\big).$$

Everything else is completely analogous to the indenter potential. You can test your element by the derivative checker.

(Coding) Problem 5: cube wall impact IBVP (10 points).

Let us solve an IBVP that exploits the new explicit dynamic solver in finite deformations (using our simplicial element and the Neo-Hookean material model). Consider the 3D cuboidal body from problem 2 (choosing $\alpha=\beta=10^{-5}$ damps vibrational noise). Let us simulate the collision of the cuboid with a (nearly) rigid wall approximated by the potential from problem 4 above, starting at a distance of d with an initial velocity of v. Simulate the dynamic behavior of the cube as it collides with and bounces off the wall.

Hint: Paraview lets you visualize cuts through the body, so you can inspect interior stress and strain distributions, etc. For example moduli and parameters, see the table on page 3.

(Coding) Problem 6: mystery impact IBVP (5 points).

You are given the mesh of a realistic 3D body. Repeat the wall impact simulation with this mesh and have fun playing with parameters (e.g., changing α,β you can observe the influence of vibration damping).

total: 100 points

Happy Thanksgiving!

Parameters for Problem 5

You are welcome to adjust the simulation parameters; the following are suggested values that lead to reasonable results:

totalTime $= 0.08$ *(total simulated time)*
solverTimeStep $= 2 \cdot 10^{-4}$ *(Δt in the simulation)*
initialVelocity $v = -10$ *(initial vertical velocity of the cube)*
shearModulus $\mu = 5 \cdot 10^5$
bulkModulus $\kappa = 5 \cdot 10^7$
density $= \rho = 1522$
wallForceConstant $k = 1 \cdot 10^8$
dampingAlpha $\alpha = 1 \cdot 10^{-5}$
dampingBeta $\beta = 1 \cdot 10^{-5}$
distanceAboveWall $d = 0$ *(initial distance of the cube to the wall)*